MATH 1A - MOCK MIDTERM 2 - SOLUTIONS

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1. *(15 points)* Using **the definition** of the derivative, find $f'(1)$, where $f(x) = \frac{1}{x}$.

$$
\begin{aligned}
f'(1) &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\
&= \lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1} \\
&= \lim_{x \to 1} \frac{\frac{1-x}{x}}{x - 1} \\
&= \lim_{x \to 1} \frac{1 - x}{x(x-1)} \\
&= \lim_{x \to 1} \frac{-(x-1)}{x(x-1)} \\
&= \lim_{x \to 1} \frac{-1}{x} \\
&= -1
\end{aligned}
$$

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2. *(15 points)* Using **the definition** of the derivative, calculate the derivative of $f(x) = \sqrt{x} + x$

$$\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{x + \sqrt{x} - (a + \sqrt{a})}{x - a} \\
&= \lim_{x \to a} \frac{x + \sqrt{x} - a - \sqrt{a})}{x - a} \\
&= \lim_{x \to a} \frac{(x - a) + (\sqrt{x} - \sqrt{a})}{x - a} \\
&= \lim_{x \to a} \frac{x - a}{x - a} + \frac{\sqrt{x} - \sqrt{a}}{x - a} \\
&= \lim_{x \to a} \frac{x - a}{x - a} + \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \\
&= \lim_{x \to a} 1 + \lim_{x \to a} \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{(x - a)(\sqrt{x} + \sqrt{a})} \\
&= 1 + \lim_{x \to a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} \\
&= 1 + \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} \\
&= 1 + \frac{1}{\sqrt{a} + \sqrt{a}} \\
&= 1 + \frac{1}{2\sqrt{a}}
\end{aligned}$$

Hence, $\boxed{f'(x) = 1 + \frac{1}{2\sqrt{x}}}$

3. *(50 points, 5 points each)* Find the derivatives of the following functions:

(a) $f(x) = \frac{x+e^x}{e^x+1}$

$$f'(x) = \frac{(1+e^x)(e^x+1) - (x+e^x)(e^x)}{(e^x+1)^2}$$

(b) $f(x) = -\tan^{-1}(\frac{1}{x})$

$$f'(x) = -\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right) = \frac{1}{x^2(1+\left(\frac{1}{x}\right)^2)} = \frac{1}{x^2+1}$$

Note: $\tan^{-1}(x)$ and $f(x)$ have the same derivative, so in fact, as we'll see later, this means that $\tan^{-1}(x) = -\tan^{-1}(\frac{1}{x}) + C$, that is $\tan^{-1}(x) + \tan^{-1}(\frac{1}{x}) = C$. To find C, plug in $x = 1$, and you get $\tan^{-1}(1) + \tan^{-1}(1) = C$, so $\frac{\pi}{4} + \frac{\pi}{4} = C$, so $C = \frac{\pi}{2}$, and in fact, we get:

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

WOOOOOOOOW!!! :O How cool is that?

(c) $f''(x)$, where $f(x) = \sin(x)e^x$

$$f'(x) = \cos(x)e^x + \sin(x)e^x$$

$$f''(x) = -\sin(x)e^x + \cos(x)e^x + \cos(x)e^x + \sin(x)e^x = 2\cos(x)e^x$$

(d) The equation of the tangent line to $y = \frac{e^x}{x}$ at the point $(1, e)$

$$y' = \frac{e^x x - e^x}{x^2}$$

Slope = $y'(1) = \frac{e-e}{1} = 0$

Hence equation: $y - e = 0(x-1)$, so $\boxed{y = e}$

(e) $f(x) = \ln(\sqrt{x^2+1})$

$$f'(x) = \left(\frac{1}{\sqrt{x^2+1}}\right)\left(\frac{1}{2\sqrt{x^2+1}}\right)(2x) = \frac{x}{x^2+1}$$

Note: A smarter way would be to notice that $f(x) = \frac{1}{2}\ln(x^2+1)$ (by properties of $\ln$)

(f) $f(x) = \ln(\ln(\ln(x)))$

$$f'(x) = \left(\frac{1}{\ln(\ln(x))}\right)\left(\frac{1}{\ln(x)}\right)\left(\frac{1}{x}\right)$$

(g) y' where $x^2 + xy + y^2 = 3$

$$\begin{aligned} 2x + y + xy' + 2yy' &= 0 \\ y'(x+2y) &= -(2x+y) \\ y' &= -\frac{2x+y}{x+2y} \end{aligned}$$

(h) $f(x) = x^{\cos(x)}$

<u>Logarithmic differentiation:</u>

1) $y = x^{\cos(x)}$

2) $\ln(y) = \cos(x)\ln(x)$

3) $\frac{y'}{y} = -\sin(x)\ln(x) + \cos(x)\frac{1}{x}$

4) $y' = y\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right) = x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right)$

(i) y' at $(0, -2)$, where $y^2(y^2 - 4) = x^2(x^2 - 5)$

$$\begin{aligned}
2yy'(y^2-4) + y^2(2yy') &= 2x(x^2-5) + x^2(2x) \\
2(-2)y'((-2)^2-4) + (-2)^2(2(-2)y') &= 2(0)(0^2-5) + 0^2(2\times 0) \\
-4y'(0) + 4(-4)y' &= 0 \\
-16y' &= 0 \\
y' &= 0
\end{aligned}$$

(j) y', where $x^y = y^x$ (**Hint:** Take $\ln$'s)

$$\begin{aligned}
\ln(x^y) &= \ln(y^x) \\
y\ln(x) &= x\ln(y) \\
y'\ln(x) + y\frac{1}{x} &= \ln(y) + x\frac{y'}{y} \\
y'\ln(x) + \frac{y}{x} &= \ln(y) + \frac{xy'}{y} \\
y'\left(\ln(x) - \frac{x}{y}\right) &= \ln(y) - \frac{y}{x} \\
y' &= \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}
\end{aligned}$$

4. *(20 points)* Remember that one of the following problems will be for sure on your exam:

Problem 1: Show that the equation of the tangent line to the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

Slope:

$$\begin{aligned}
\frac{2x}{a^2} + \frac{2yy'}{b^2} =& 0\\
y'\left(\frac{2y}{b^2}\right) =& -\frac{2x}{a^2}\\
y' =& -\frac{b^2}{a^2}\frac{2x}{2y}\\
y' =& -\frac{b^2}{a^2}\frac{x}{y}
\end{aligned}$$

Equation:
At (x_0, y_0), the slope is $-\frac{b^2}{a^2}\frac{x_0}{y_0}$, so the equation of the tangent line at (x_0, y_0) is:

$$y - y_0 = \left(-\frac{b^2}{a^2}\frac{x_0}{y_0}\right)(x - x_0)$$

Simplification:
First of all, by multiplying both sides by $a^2 y_0$, we get:

$$(y - y_0)(a^2 y_0) = -b^2 x_0 (x - x_0)$$

Expanding out, we get:

$$y a^2 y_0 - a^2 (y_0)^2 = -b^2 x_0 x + b^2 (x_0)^2$$

Now rearranging, we have:

$$ya^2y_0 + b^2x_0x = a^2(y_0)^2 + b^2(x_0)^2$$

Now dividing both sides by a^2, we get:

$$yy_0 + \frac{b^2}{a^2}x_0x = (y_0)^2 + \frac{b^2}{a^2}(x_0)^2$$

And dividing both sides by b^2, we get:

$$\frac{yy_0}{b^2} + \frac{x_0x}{a^2} = \frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2}$$

But now, since (x_0, y_0) is on the ellipse, $\frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2} = 1$, we get:

$$\frac{yy_0}{b^2} + \frac{x_0x}{a^2} = 1$$

Whence,

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

Problem 2: Show that the equation of the tangent line to the hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$$

This is super similar to Problem 1:
Slope:

$$\begin{aligned} \frac{2x}{a^2} - \frac{2yy'}{b^2} =& 0 \\ y'\left(-\frac{2y}{b^2}\right) =& -\frac{2x}{a^2} \\ y' =& \frac{b^2}{a^2}\frac{2x}{2y} \\ y' =& \frac{b^2}{a^2}\frac{x}{y} \end{aligned}$$

Equation:
At (x_0, y_0), the slope is $\frac{b^2}{a^2}\frac{x_0}{y_0}$, so the equation of the tangent line at (x_0, y_0) is:

$$y - y_0 = \left(\frac{b^2}{a^2}\frac{x_0}{y_0}\right)(x - x_0)$$

Simplification:
First of all, by multiplying both sides by $a^2 y_0$, we get:

$$(y - y_0)(a^2 y_0) = b^2 x_0 (x - x_0)$$

Expanding out, we get:

$$y a^2 y_0 - a^2 (y_0)^2 = b^2 x_0 x - b^2 (x_0)^2$$

Now rearranging, we have:

$$ya^2y_0 - b^2x_0x = a^2(y_0)^2 - b^2(x_0)^2$$

Now dividing both sides by a^2, we get:

$$yy_0 - \frac{b^2}{a^2}x_0x = (y_0)^2 - \frac{b^2}{a^2}(x_0)^2$$

And dividing both sides by b^2, we get:

$$\frac{yy_0}{b^2} - \frac{x_0x}{a^2} = \frac{(y_0)^2}{b^2} - \frac{(x_0)^2}{a^2}$$

But now, since (x_0, y_0) is on the hyperbola, $\frac{(x_0)^2}{a^2} - \frac{(y_0)^2}{b^2} = 1$, so $\frac{(y_0)^2}{b^2} - \frac{(x_0)^2}{a^2} = -1$, and we get:

$$\frac{yy_0}{b^2} - \frac{x_0x}{a^2} = -1$$

Whence,

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

Problem 3: Show that the sum of the $x-$ and $y-$ intercepts of any tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is equal to c.

Slope:

$$\begin{aligned} \frac{1}{2\sqrt{x}} + y'\left(\frac{1}{2\sqrt{y}}\right) =& 0 \\ y'\left(\frac{1}{2\sqrt{y}}\right) =& -\frac{1}{2\sqrt{x}} \\ y' =& -\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}} \\ y' =& -\frac{2\sqrt{y}}{2\sqrt{x}} \\ y' =& -\frac{\sqrt{y}}{\sqrt{x}} \end{aligned}$$

Equation: At (x_0, y_0), the slope is $-\frac{\sqrt{y_0}}{\sqrt{x_0}}$, and so the equation of the tangent line at (x_0, y_0) is:

$$y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$$

$y-$intercept:
To find the $y-$intercept, set $x = 0$ and solve for y:

$$\begin{aligned} y - y_0 =& -\frac{\sqrt{y_0}}{\sqrt{x_0}}(0 - x_0) \\ y - y_0 =& -\frac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0) \\ y - y_0 =& \sqrt{y_0}\sqrt{x_0} \\ y =& y_0 + \sqrt{y_0}\sqrt{x_0} \end{aligned}$$

$x-$intercept:
To find the $x-$intercept, set $y = 0$ and solve for x:

$$
\begin{aligned}
0 - y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \\
-y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \\
x - x_0 &= -\frac{\sqrt{x_0}}{\sqrt{y_0}}(-y_0) \\
x &= x_0 + \sqrt{x_0}\sqrt{y_0}
\end{aligned}
$$

<u>Sum:</u>
The sum of the $y-$ and $x-$ intercepts is:

$$(y_0 + \sqrt{y_0}\sqrt{x_0}) + (x_0 + \sqrt{x_0}\sqrt{y_0}) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0$$

But the trick is that:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0})^2 + 2\sqrt{x_0}\sqrt{y_0} + (\sqrt{y_0})^2 = (\sqrt{x_0} + \sqrt{y_0})^2$$

But since (x_0, y_0) is on the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$, we get $\sqrt{x_0} + \sqrt{y_0} = \sqrt{c}$.

And so, finally we get that the sum of the $x-$ and $y-$ intercepts is:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2 = (\sqrt{c})^2 = c$$

Bonus 1 *(5 points)* Give examples of functions f and g with $f(x) \leq g(x)$, but:

(i) $f'(x) \leq g'(x)$
(ii) $f'(x) = g'(x)$
(iii) $f'(x) \geq g'(x)$

Hint: If you want to, one of your functions can be the zero function!

Note: This problem is meant to show you that derivatives can behave in very strange ways. We've seen that this is not the case with limits, i.e. if $f(x) \leq g(x)$, $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$, and we will also see that this is not the case with integrals, i.e. if $f(x) \leq g(x)$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. There is more excitement to come on the actual midterm :)

(i) $f(x) = 0$, $g(x) = e^x$

(ii) $f(x) = x+1$, $g(x) = x+2$ ($f(x) = 0$ and $g(x) = 1$ also works)

(iii) $f(x) = 0$, $g(x) = \frac{1}{x^2}$

Bonus 2 *(5 points)*

(a) Let

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Does $f'(0)$ exist?

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x\sin(\frac{1}{x}) - 0}{x} = \lim_{x\to 0} \sin(\frac{1}{x})$$

As the last limit does not exist, $\boxed{f'(0) \text{ DNE}}$

(b) What about

$$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2\sin(\frac{1}{x}) - 0}{x} = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$$

(by the squeeze theorem)